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FOLIATED MANIFOLDS WITH FLAT BASIC CONNECTION

ROBERT A. BLUMENTHAL

1. Introduction and statement of results

Let $\mathcal{F}$ be a smooth codimension-q foliation of a smooth manifold M. Let $T(M)$ denote the tangent bundle of M, and let $E \subset T(M)$ be the subbundle consisting of the vectors tangent to the leaves of $\mathcal{F}$. Let $Q = T(M)/E$ be the normal bundle of $\mathcal{F}$, and let $F(Q)$ be its frame bundle, a principal $GL(q, R)$ bundle. Recall that a connection on $F(Q)$ is said to be basic if the parallel translation which it defines along paths lying in a leaf of $\mathcal{F}$ agrees with the "natural parallelism along the leaves" [3]. Equivalently, if π: $T(M) \to Q$ is the natural projection, and if $\Gamma(E)$, $\Gamma(Q)$, and $\mathfrak{X}(M)$ denote the space of smooth sections of the vector bundles E, Q, and $T(M)$ respectively, then the associated Koszul operator ∇: $\mathfrak{X}(M) \times \Gamma(Q) \to \Gamma(Q)$ satisfies the condition that $\nabla_X Y = \pi([X, \tilde{Y}])$ for all $X \in \Gamma(E)$ and all $Y \in \Gamma(Q)$, where $\tilde{Y}$ is any vector field on M such that $\pi(\tilde{Y}) = Y$, and $[X, \tilde{Y}]$ denotes the usual Lie bracket of vector fields [2]. In the present work we study foliated manifolds supporting a flat basic connection, that is, a basic connection with vanishing curvature and torsion.

To begin, we have the following nonexistence result.

Theorem 1. *If M is compact with finite fundamental group, then M does not support a foliation with flat basic connection.*

As a corollary to the proof of Theorem 1, we will obtain

Corollary 1. *Let $(M, \mathcal{F})$ be a foliated manifold with flat basic connection. If $H_1(M, Z) = 0$, then $\mathcal{F}$ admits a transverse volume element; that is, $\mathcal{F}$ is defined by a nowhere zero closed q-form on M, $q = \mathrm{codim}(\mathcal{F})$.*

It is well-known (see, e.g., [6]) that the universal cover of an n-dimensional manifold supporting a complete flat linear connection is R^n where the lifted connection corresponds to the canonical linear connection on R^n. We generalize this codimension-n result to foliations of arbitrary codimension.

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Theorem 2. *Let $(M, \mathcal{F})$ be a foliated manifold with a complete flat basic connection. Then the universal cover $\tilde{M}$ of M is a product $\tilde{L} \times R^q$, where $\tilde{L}$ is the (common) universal cover of the leaves of $\mathcal{F}$, the leaves of the lifted foliation are identified with the sets $\tilde{L} \times \{x\}$, $x \in R^q$, and the lifted connection corresponds to the basic connection on $\tilde{L} \times R^q$ determined by the canonical linear connection on R^q.*

Corollary 2. *If M^n supports a nonsingular flow with a complete flat basic connection, then the universal cover of M^n is R^n.*

Corollary 3. *Let $(M^n, \mathcal{F})$ be a codimension-$(n-2)$ foliation with a complete flat basic connection. Then either*

(i) *the universal cover of M^n is R^n, or*

(ii) *the leaves of $\mathcal{F}$ are spheres and projective planes.*

Theorem 3. *Let $\mathcal{F}$ be a codimension-one foliation of a compact manifold M with a complete flat basic connection. Then either*

(i) *all the leaves of $\mathcal{F}$ are dense, or*

(ii) *all the leaves of $\mathcal{F}$ have polynomial growth of degree $\leqslant \beta_1(M)$, the first Betti number of M.*

In particular, $\mathcal{F}$ has no exceptional minimal sets.

2. Proofs of the theorems

Let $(M, \mathcal{F})$ be a foliated manifold with a flat basic connection. Via a choice of Riemannian metric on M, we may regard Q as a subbundle of $T(M)$ complementary to E. Thus $T(M) = E \oplus Q$, and the covariant differentiation operator ∇ corresponding to the basic connection then satisfies

$$\nabla_X Y = [X, Y]_Q \quad \text{for all } X \in \Gamma(E),\ Y \in \Gamma(Q),$$

where $[X, Y]_Q$ denotes the Q-component of the Lie bracket of the vector fields X and Y.

Let p: $F(Q) \to M$ be the bundle projection. The connection on $F(Q)$ gives rise to a smooth $GL(q, R)$-invariant distribution H on $F(Q)$ such that $T(F(Q)) = V \oplus H$ where $V \subset T(F(Q))$ is the subbundle consisting of vertical vectors, i.e., vectors tangent to the fibers of p. Let ω be the corresponding connection form, a smooth $gl(q, R)$-valued one-form on $F(Q)$. The curvature form is the $gl(q, R)$-valued two-form Ω on $F(Q)$ defined by $\Omega_u(X, Y) = (d\Omega)_u(X_H, Y_H)$, $u \in F(Q)$, $X, Y \in T_u(F(Q))$ where X_H and Y_H are the H-components of X and Y respectively. For $u \in F(Q)$, $X \in T_u(F(Q))$, let $\theta_u(X)$ be the ordered q-tuple of real numbers obtained by taking the components of the vector $(p_{*u}(X))_Q$ with respect to the basis u of $Q_{p(u)}$. Then θ is a smooth R^q-valued one-form on

$F(Q)$. The torsion form of H is the R^q-valued two-form Θ on $F(Q)$ defined by

$$\Theta_u(X, Y) = (d\theta)_u(X_H, Y_H), \quad u \in F(Q), X, Y \in T_u(F(Q)).$$

Since H is flat, we have $\Omega = \Theta = 0$.

Let $(\omega^i_j)^q_{i,j=1}$ and $(\Omega^i_j)^q_{i,j=1}$ be the components of ω, respectively Ω, with respect to the standard basis of $gl(q, R)$. Let $(\theta^i)^q_{i=1}$ and $(\Theta^i)^q_{i=1}$ be the components of θ, respectively Θ, with respect to the standard basis of R^q. Since $\Theta^i = 0$ for $i = 1,\cdots,q$ and $\Omega^i_j = 0$ for $i, j = 1,\cdots,q$, the structure equations of the connection take the form

$$d\theta^i = -\sum_j \omega^i_j \wedge \theta^j, \quad i = 1,\cdots,q$$

$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j, \quad i,j = 1,\cdots,q.$$

Let $h \in R^q$. For each $u \in F(Q)$, let $B(h)_u$ be the unique horizontal vector in $T_u(F(Q))$ such that $p_{*u}(B(h)_u) = h_1u_1 + \cdots + h_qu_q$ where $h = (h_1,\cdots,h_q)$, $u = (u_1,\cdots,u_q)$. This defines the basic vector field $B(h)$ on $F(Q)$ corresponding to h. Clearly $\theta(B(h)) \equiv h$ for all $h \in R^q$. Let $\{e_1,\cdots,e_q\}$ be the standard basis of R^q, and $B(e_1),\cdots,B(e_q)$ the corresponding basic vector fields.

Let $x \in M$ and $u \in p^{-1}(x)$. Since $\Omega = 0$, the distribution H is integrable, and hence we can find a neighborhood U of x in M and a smooth section s: $U \to F(Q)$ such that $s(U)$ is an integral manifold of H. For $y \in U$, set $X_{i_y} = p_*(B(e_i)_{s(y)})$, $i = 1,\cdots,q$. Then $X_1,\cdots,X_q$ are smooth independent normal vector fields on U. We have

$$\begin{aligned} 0 = \Theta\big(B(e_i), B(e_j)\big) &= d\theta\big(B(e_i), B(e_j)\big) \\ &= B(e_i)\theta\big(B(e_j)\big) - B(e_j)\theta\big(B(e_i)\big) - \theta\big(\big[B(e_i), B(e_j)\big]\big) \\ &= -\theta\big(\big[B(e_i), B(e_j)\big]\big), \end{aligned}$$

and so $[X_i, X_j]_Q = 0$. Since $X_1,\cdots,X_q$ are parallel with respect to the connection H, and hence parallel along the leaves of $\mathcal{F}$, there exists (shrinking U if necessary) a smooth submersion f: $U \to R^q$ such that kernel $(f_{*_y}) = E_y$ and

$$f_{*_y}\big(X_{i_y}\big) = \frac{\partial}{\partial x^i}\bigg|_{f(y)}, \quad i = 1,\cdots,q \quad \text{for all } y \in U.$$

Let $F(R^q)$ be the frame bundle of R^q, and ω' be the connection form on $F(R^q)$ corresponding to the canonical linear connection on R^q. Let f_*: $p^{-1}(U) \to F(R^q)$ be the map induced by f. Since H is a basic connection for $\mathcal{F}$, it follows that the foliation of $p^{-1}(U)$ whose leaves are the level sets of f_* is horizontal. Thus we have decompositions

(1) $H = \text{kernel}\,(f_*)_* \oplus \text{span}\{B(e_1),\cdots,B(e_q)\}$,

(2) $T(F(Q)) = V \oplus \text{kernel}(f_*)_* \oplus \text{span}\{B(e_1),\cdots,B(e_q)\}$.
Since ω and $(f_*)^*\omega'$ agree on each of the subbundles occurring in (2), we have that $\omega = (f_*)^*\omega'$ on $p^{-1}(U)$. Thus we can choose an R^q-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ on M where

(i) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M;

(ii) f_α: $U_\alpha \to R^q$ is a smooth submersion constant along the leaves of $\mathfrak{F}/U_\alpha$;

(iii) $g_{\alpha\beta}$: $f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism satisfying $f_\alpha = g_{\alpha\beta} \circ f_\beta$ on $U_\alpha \cap U_\beta$

such that $(f_{\alpha_*})^*\omega' = \omega$ on $p^{-1}(U_\alpha)$ for each $\alpha \in A$.

If $U_\alpha \cap U_\beta \neq \varnothing$, then we have $(f_{\beta_*})^*(g_{\alpha\beta_*})^*\omega' = (g_{\alpha\beta} \circ f_\beta)^*_*\omega' = (f_{\alpha_*})^*\omega' = \omega = (f_{\beta_*})^*\omega'$. Hence $(g_{\alpha\beta_*})^*\omega' = \omega'$ on $F(R^q)\,|_{f_\beta(U_\alpha\cap U_\beta)}$, and so $g_{\alpha\beta}$ is the restriction of an affine transformation of R^q. Let π: $\tilde{M} \to M$ be the universal cover of M. There exists a submersion f: $\tilde{M} \to R^q$ constant along the leaves of $\tilde{\mathfrak{F}} = \pi^{-1}(\mathfrak{F})$ [1]. This is clearly impossible if M is compact with finite fundamental group thus proving Theorem 1.

Let G be the group of affine transformations of R^q, that is, the semi-direct product of R^q and $GL(q, R)$. By [1], there is a homomorphism Φ: $\pi_1(M) \to G$ such that for each covering transformation $\tau \in \pi_1(M)$ the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{f} & R^q \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\Phi(\tau)} \\ \tilde{M} & \xrightarrow[f]{} & R^q \end{array}$$

is commutative. Let ρ: $\pi_1(M) \to R$ be the composition

$$\pi_1(M) \xrightarrow{\Phi} G \xrightarrow{\alpha} GL(q, R) \xrightarrow{\det} R$$

where α is projection onto the $GL(q, R)$ factor, and det denotes the determinant function. If $H_1(M, Z) = 0$, then ρ is the trivial homomorphism, and hence the image of Φ is contained in the subgroup of G given by the semi-direct product of R^q and $SL(q, R)$. Thus we can find an R^q-cocycle $\{(U'_\alpha, f'_\alpha, g'_{\alpha\beta})\}_{\alpha,\beta\in A'}$ defining $\mathfrak{F}$ such that each $g'_{\alpha\beta}$ preserves the natural volume element on R^q. This induces a nowhere zero closed q-form on M defining $\mathfrak{F}$.

Suppose now that H is complete. Then H lifts to a complete flat basic connection $\tilde{H}$ on the bundle of normal frames of $\tilde{\mathfrak{F}}$. Since $\tilde{M}$ is simply connected, the holonomy group of $\tilde{H}$ is trivial and hence $\tilde{\mathfrak{F}}$ is a transversely complete e-foliation [3]. Thus the leaf space $\tilde{M}/\tilde{\mathfrak{F}}$ is a smooth Hausdorff q-dimensional manifold, and the natural projection $\tilde{M} \to \tilde{M}/\tilde{\mathfrak{F}}$ is a smooth fiber bundle whose fibers are the leaves of $\tilde{\mathfrak{F}}$ [3], [4]. Let $\tilde{\nabla}$ be the covariant differentiation operator arising from the connection $\tilde{H}$. Let X and Y be smooth vector fields on $\tilde{M}/\tilde{\mathfrak{F}}$. Let $\tilde{X}$ and $\tilde{Y}$ be smooth normal vector fields on $\tilde{M}$ which are parallel along the leaves of $\tilde{\mathfrak{F}}$ and project to X and Y respectively. Then if $\tilde{Z}$ is a smooth vector field on $\tilde{M}$ tangent to the leaves of $\tilde{\mathfrak{F}}$, the vanishing of the curvature of $\tilde{H}$ gives $\tilde{\nabla}_{\tilde{Z}}\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Z}}\tilde{Y} + \tilde{\nabla}_{[\tilde{Z},\tilde{X}]}\tilde{Y}$. But $[\tilde{Z}, \tilde{X}]$ is tangent to $\tilde{\mathfrak{F}}$ since $\tilde{X}$ is parallel along the leaves. Hence, since $\tilde{Y}$ is parallel along the leaves, we have $\tilde{\nabla}_{\tilde{Z}}\tilde{Y} = \tilde{\nabla}_{[\tilde{Z},\tilde{X}]}\tilde{Y} = 0$. Thus $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ is parallel along the leaves of $\tilde{\mathfrak{F}}$, and hence projects to a vector field $\hat{\nabla}_X Y$ on $\tilde{M}/\tilde{\mathfrak{F}}$. Clearly $\hat{\nabla}$ defines a complete flat linear connection on $\tilde{M}/\tilde{\mathfrak{F}}$ which pulls back to $\tilde{H}$ on $\tilde{M}$. Since $\tilde{M}$ is simply connected, the exact homotopy sequence of the fibration shows that $\tilde{M}/\tilde{\mathfrak{F}}$ is simply connected. Hence $\tilde{M}/\tilde{\mathfrak{F}}$ is affinely isomorphic to R^q with its canonical linear connection [6]. Since R^q is contractible, the leaves of $\tilde{\mathfrak{F}}$ are simply connected and $\tilde{\mathfrak{F}}$ is a product foliation thus completing the proof of Theorem 2.

Suppose that M is compact, and let $\mathfrak{F}$ be a codimension-one foliation of M supporting a complete flat basic connection. Let $\pi: \tilde{M} \to M$ be the universal cover of M, and $f: \tilde{M} \to R$ be a fibration whose fibers are the leaves of $\tilde{\mathfrak{F}}$. Let $G = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right): a \neq 0\}$ be the two-dimensional affine group. Let $\Gamma = \text{image } \Phi$. Then Γ is a finitely generated subgroup of G which acts in a natural way on R. For $x \in R$, let $\Gamma(x)$ denote the orbit of x under Γ. Let $L \in \mathfrak{F}$. Choose a leaf $\tilde{L} \in \tilde{\mathfrak{F}}$ such that $\pi(\tilde{L}) = L$, and let $x = f(\tilde{L})$. Then $\Gamma(x)$ depends only on the leaf L, and we denote this orbit by Γ^L. Clearly L is dense in M if and only if Γ^L is dense in R. Suppose Γ is abelian. Then Φ induces a surjection $H_1(M, Z) \to \Gamma$, and hence Γ has polynomial growth of degree $\leqslant \beta_1(M)$. Thus all the leaves of $\mathfrak{F}$ have polynomial growth of degree $\leqslant \beta_1(M)$, [1]. If Γ is not abelian, then all the orbits of Γ are dense in R, and so all the leaves of $\mathfrak{F}$ are dense. Since a leaf in an exceptional minimal set of a C^2 codimension-one foliation has exponential growth [5], it follows that $\mathfrak{F}$ has no exceptional minimal sets.

The following example shows that completeness is an essential hypothesis in Theorem 2. Define $f: R^3 \to R$ by $f(x, y, z) = e^y \sin 2\pi x$. Then f is a smooth submersion, and defines a codimension-one foliation $\tilde{\mathfrak{F}}$ of R^3. This foliation is invariant under the action of Z^3 on R^3, and hence passes to a foliation $\mathfrak{F}$ of the three-dimensional torus. Let G be the two-dimensional affine group, and define $\Phi: Z^3 \to G$ by $\Phi(n, m, p) = \left(\begin{smallmatrix} e^m & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $f \circ T_{(n,m,p)} = \Phi(n, m, p) \circ f$ for all $(n, m, p) \in Z^3$ where $T_{(n,m,p)}$ denotes the translation of R^3 determined by (n, m, p). Hence there is a Haefliger cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ defining $\mathfrak{F}$ such that each $g_{\alpha\beta}$ is the restriction of some $\Phi(n, m, p)$. The canonical linear connection on R is preserved by the maps $\Phi(n, m, p)$, and hence induces a flat basic connection for $\mathfrak{F}$. This connection however is not complete. Indeed, the leaf space of $\tilde{\mathfrak{F}}$ is a non-Hausdorff one-manifold.

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SAINT LOUIS UNIVERSITY